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THE THEORY OF TWO-DIMENSIONAL PROCESSES IN INHOMOGENEOUS LAYERS WITH A POWER LAW OF THEIR CONDUCTIVITY VARIATION†

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A theory of functions, defined by canonical equations of two-dimensional processes in inhomogeneous layers with a power law of their conductivity variation, is developed, and fundamental solutions and systems of functions based on them are obtained. The class of functions introduced and investigated possesses, in addition to general properties [1–3], properties which are characteristic solely for this class, and these are investigated. The theory developed enables boundary-value problems to be solved in closed form, which is realized in the conjunction problems considered for two-dimensional processes of different physical kinds occurring in inhomogeneous layers.

1. Consider two-dimensional steady processes in an inhomogeneous medium of thickness h, situated in a plane, where the Cartesian coordinate axes are chosen. These processes describe the quasi-potential of the velocity φ and stream function ψ, which satisfy the following system of equations, represented in dimensionless form [4]

$$\frac{\partial\varphi}{\partial\xi}=\frac{1}{P}\frac{\partial\psi}{\partial\eta},\quad \frac{\partial\varphi}{\partial\eta}=-\frac{1}{P}\frac{\partial\psi}{\partial\xi},\quad P=P(\xi,\eta)>0 \tag{1.1}$$

where $P = kh$ is the conductivity of the layer and k is its permeability.

Introducing the complex potential

$$W=\varphi+i\psi/P \tag{1.2}$$

Eqs (1.1) can be written in the following form [5] in the $\zeta=\xi=i\eta$ plane

$$\frac{\partial W}{\partial\bar\zeta}+A(\zeta,\bar\zeta)(W-\overline{W})=0 \tag{1.3}$$

$$A(\zeta,\bar\zeta)=\frac{\partial}{\partial\bar\zeta}\ln\sqrt{P(\zeta,\bar\zeta)}\quad\left(2\frac{\partial}{\partial\bar\zeta}\equiv\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\eta}\right).$$

Equation (1.3) is covariant under certain transformations of the complex potential W. We have the following transformation

$$W_1=-iPW\quad(\varphi_1=\psi,\ \psi_1=-\varphi) \tag{1.4}$$

which relates the complex potentials (1.2) and $W_1=\varphi_1+i\psi_1/P$ of one and the same process in conjugate [6] layers of conductivities P and $P_1=1/P$. Equation (1.4) enables us, on the basis of processes investigated in a certain layer, to investigate the same processes in a layer conjugate to it.

We will consider a layer with a power law of conductivity variation

$$P=f^s(\xi,\ \eta),\ s>0 \tag{1.5}$$

($f(\xi,\eta)$ is an harmonic function of the coordinates), for which Eq. (1.3) is reduced to canonical form. We will use the conformal covariance of Eq. (1.3) and choose the conformal transformation $z=F(\zeta)$ ($y=\operatorname{Im}F(\zeta)=f(\xi,\eta)$) connecting the planes $z=x+iy$ and ζ. With this transformation, the boundary

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$f(\xi, \eta) = 0$ of the region of the process in the ζ plane is transformed into the straight line $y = 0$. The process will occur in the half-plane $z(\operatorname{Im} z > 0)$ and is described in it by the canonical equation which follows from (1.3)

$$\frac{\partial W}{\partial \bar{z}} - \frac{s(W - \overline{W})}{2(z - \bar{z})} = 0 \quad \left(2\frac{\partial}{\partial \bar{z}} \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{1.6}$$

Hence, a study of the processes in conductivity layer (1.5) and also in accordance with (1.4) in the layer conjugate to it ($s < 0$) reduces to finding solutions of Eq. (1.6).

Equation (1.6) is a complex representation of the canonical system of equations which follows from (1.1) when $\xi = x$, $\eta = y$ and $P = y^s$. It describes two-dimensional processes in a layer of conductivity $P = y^s$ (in particular, when $s = 1$, spatial axisymmetric processes). This layer has a singular line $y = 0$ (the x axis), on which $P = 0$ and, consequently, Eq. (1.6) is degenerate.

When $s = 0$, Eq. (1.6) takes the form of the Cauchy–Riemann equation $\partial W/\partial z = 0$, which is satisfied by analytic functions describing plane-parallel processes in homogeneous layers ($P = 1$). Consequently, analytic functions are a special case (when $s = 0$) of the functions $W(z)$, which satisfy Eq. (1.6).

Equation (1.6) is a special case of the general form of an equation [1], the coefficients of which belong, in the region D in which they are defined, to the set $L_p(D)$ of functions summed with power $p > 2$. The coefficients $\pm s(z - \bar{z})^{-1/2}$ of Eq. (1.6) may belong to the set $L_p(D)$, $p > 2$ only in those regions D of the half-plane $\operatorname{Im} z \geqslant 0$ which do not contain points of the singular line $y = 0$. In these regions the earlier properties of the functions [1], which satisfy equations of general form, are extended to the solution $W(z)$ of Eq. (1.6). However, the solutions of Eq. (1.6) possess a number of properties that are characteristic solely of them, due to the presence of the singular line $y = 0$.

2. One such property is the transformation of the inversion of the function $W(z)$ with respect to a semicircle (a sphere when $s = 1$) of radius a, carried out from the origin of coordinates. This transformation is as follows: if the function $W_0(z)$ satisfies Eq. (1.6), its solution will be the function

$$W(z) = \left(\frac{a}{|z|}\right)^s \left\{\overline{W_0\left(\frac{a^2}{\bar{z}}\right)} - \frac{s}{2}\int_p^1 \tau^{\sigma s - 1}\left[W_0\left(\frac{a^2\tau}{\bar{z}}\right) + (2\sigma - 1)\overline{W_0\left(\frac{a^2\tau}{\bar{z}}\right)}\right]d\tau\right. \tag{2.1}$$

where the values of p and σ are defined by the form of $W_0(z)$. That is, if the singular points of the function $W_0(z)$ lie in the region $D_1(|z| > a)$ and $|W_0(z)| = O(|z|^{\mu_1})$ as $|z| \to 0$, then $p = 0$, $\sigma s = \mu - \mu_1$; if its singular points lie in the region $D_2(|z| < a)$ and $|W_0(z)| = O(|z|^{-\mu_2})$ as $|z| \to \infty$, then $p = \infty$, $\sigma s = \mu_2 - \mu$, $\mu = \text{const} > 0$. If the function $W_0(z)$ has singular points in the region D_1 (or D_2), the singular points of the function $W(z)$ will be in the region D_2 (or D_1).

By finding the functions φ and ψ by means of (1.2) and (2.1) we can verify that they satisfy Eqs (1.1) when $\xi = x$, $\eta = y$ and $P = y^s$, $s > 0$. Consequently, (2.1) is the solution of Eq. (1.6).

In particular, transformations of the inversion of the function $W(z)$ with respect to a sphere (for $s = 1$) [7] and a circle (for $s = 0$) follow from (2.1): $W(z) = \overline{W_0(a^2/\bar{z})}$, characteristic for analytic functions.

For the function $W(z)$, unique and continuous together with the partial derivatives with respect to x and y in the region D of the half-plane $\operatorname{Im} z \geqslant 0$, we will introduce [3, 4] operations of differentiation and integration. We define a Σ-derivative $W_\Sigma \equiv d_\Sigma W/dz$ such that it satisfies Eq. (1.6) everywhere in the region D, i.e.

$$W_\Sigma \equiv \frac{d_\Sigma W}{dz} = \frac{\partial W}{\partial z} + \frac{s(W - \overline{W})}{2(z - \bar{z})} = \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y} = \frac{1}{y^s}\left(\frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}\right) \tag{2.2}$$

and introduce the Σ-integral of $W(z)$ along the piecewise-smooth contour ($C \in D$) as follows:

$$\int_C W(\zeta)d_\Sigma\zeta = \frac{1}{2}\int_C \left[1 + \left(\frac{\zeta - \bar{\zeta}}{z - \bar{z}}\right)^s\right]W(\zeta)d\zeta + \left[1 - \left(\frac{\zeta - \bar{\zeta}}{z - \bar{z}}\right)^s\right]\overline{W(\zeta)d\zeta} =$$

$$= \int_C \varphi(\xi, \eta)d\xi - \psi(\xi, \eta)\eta^{-s}d\eta + iy^{-s}\int_C \psi(\xi, \eta)d\xi + \varphi(\xi, \eta)\eta^s d\eta \tag{2.3}$$

where $\zeta = \xi + i\eta$ is the variable of integration and $z, \zeta \in D$.

The Σ-integral (2.3) that does not depend on the choice of the contour C and the position of its initial point z_0 and its final point z, defines the function $W^*(z)$ as follows:

$$\int_{z_0}^{z} W(\zeta)d_\Sigma\zeta = W^*(z) - W_0^*, \quad W_0^* = \varphi^*(z_0) + i\left(\frac{2i}{z-\bar{z}}\right)^s \psi^*(z_0) \tag{2.4}$$

(W_0^* is a generalized complex constant), which is the analogue of the Newton–Leibnitz formula for analytic functions. Here $W^*(z)$ is the inverse image of the function $W(z)$: $W_\Sigma^*(z) = W(z)$.

If the contour C is closed (the points z_0 and z of this contour coincide) and it lies wholly in the simply connected region D, we have from (2.4)

$$\int_C W(\zeta)d_\Sigma\zeta = 0 \tag{2.5}$$

which is an extension of Cauchy's theorem for analytic functions to the case of the functions $W(z)$.

The Σ-derivative (2.2) and the Σ-integral (2.3) can be given a certain physical meaning, for example, in the case of the two-dimensional flow of a fluid in a layer of permeability $k = 1$ and thickness $h = y^s$, $s > 0$. That is, the Σ-derivative of the complex potential W of the flow is equal to its complex-conjugate velocity $\bar{V} = v_x - iv_y$ (v_x and v_y are the projections of the velocity onto the x and y axes): $W_\Sigma = \bar{V}$; the Σ-integral of $\bar{V}$

$$\int_C \bar{V}(\zeta)d_\Sigma\zeta = \Gamma + iy^{-s}\Pi \tag{2.6}$$

defines the circulation Γ and the flux Π of the fluid velocity vector, calculated for the contour C. Then, (2.4) for the flow velocity $\bar{V}^*(z)$ enables its complex potential $W^*(z)$ to be found, which is defined by the circulation Γ and the flux Π, calculated for the contour C, connecting the points z_0 and z. By formulating the generalized Cauchy theorem for the flow velocity $\bar{V}(z)$, we can assert, from (2.5) and (2.6), that this expresses the potentiality of the flow ($\Gamma = 0$) and the absence of a volume discharge of the liquid ($\Pi = 0$) in the simply connected region, where there are no vortices and no sources (sinks).

Note that the operations of Σ-differentiation (2.2) and Σ-integration (2.3) of the function $W(z)$ can be extended to regions including the singular line $y = 0$, on which differentiation and integration of the function $\varphi = \operatorname{Re} W(z)$ with respect to the variable x occurs. The velocity of the process on the singular line is directed along it ($v_y = 0$), and is a streamline.

3. To construct a theory of the functions $W(z)$, the so-called [2] correct fundamental solutions of Eq. (1.6)

$$F_k(z, z_0) = \Phi_k(z, z_0) + i\left(\frac{2i}{z-\bar{z}}\right)^s \Psi_k(z, z_0), \quad k = 1, 2 \tag{3.1}$$

are of fundamental importance. The functions $\Phi_k(z, z_0)$ and $\Psi_k(z, z_0)$ in the variable z satisfy Eqs (1.1), written in terms of x and y for $P = y^s$, $s > 0$.

Using well-known results [8], we obtain

$$\Phi_1(z, z_0) = -(yy_0)^{-s/2} Q_{s/2-1}(1+2\varepsilon), \quad \Psi_1(z, z_0) = \int y^s\left(\frac{\partial\Phi_1}{\partial x}dy - \frac{\partial\Phi_1}{\partial y}dx\right) \tag{3.2}$$

$$\varepsilon = \frac{R^2}{4yy_0}, \quad R = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

where $Q_{s/2-1}$ is a Legendre function of the second kind.

We will use the transition formula [6] relating the stream function to the velocity potential ψ_s in a layer of conductivity $P = y^s$ with exponents s and $s + 2$, which we write as follows:

$$\psi_s = -(yy_0)^{s+1} \varphi_{s+2} \tag{3.3}$$

As φ_{s+2} we will take from (3.2) the function $\Phi_1(z, z_0)$, in which we replace s by $s + 2$. Then, by (3.3) and Eqs (1.1) in x, y coordinates we have

$$\Psi_2(z,\ z_0)=(yy_0)^{s/2}Q_{s/2}(1+2\varepsilon),\quad \Phi_2(z,\ z_0)=\int y^{-s}\left(\frac{\partial\Psi_2}{\partial y}dx-\frac{\partial\Psi_2}{\partial x}dy\right) \tag{3.4}$$

Note that an integral representation was obtained in [8] for $\Psi_2(z, z_0)$ in terms of Bessel functions, that is only suitable when $x \geqslant 0$.

We can give the solutions (3.1), (3.2) and (3.4) a certain meaning for flows in a layer of constant permeability $k = 1$ and thickness $h = y^s$, $s > 0$. That is, the following complex potentials correspond to these solutions

$$W=\alpha F_1(z,\ z_0) \tag{3.5}$$

$$W=\beta\left(\frac{2i}{z_0-\bar{z}_0}\right)^s F_2(z,\ z_0) \tag{3.6}$$

where α, β are real constants. Taking into account the fact that the Legendre function $Q_\nu(1 + 2\varepsilon)$ has an isolated logarithmic-type singularity at the point $z_0 = x_0 + iy_0$, according to expansion (3.9(7)) in [9], following the well-known approach [6] we calculate the flux and circulation of the vector fluid velocity. We obtain $\alpha = \Pi/(2\pi)$ and $\beta = \Gamma'/(2\pi)$. Consequently, the complex potentials (3.5) and (3.6) describe a point source (or sink) of total power $\Pi > 0$ (or $\Pi < 0$) and a vortex of overall intensity Γ', directed in an anticlockwise direction ($\Gamma' > 0$) or in a clockwise direction ($\Gamma' < 0$).

Using the fundamental solutions $F_k(z, z_0)$ ($k = 1, 2$), we obtain systems of solutions of Eq. (1.6). From (3.5) and (3.6) we obtain the solution

$$L(\alpha,\ \beta,\ z,\ z_0)=\alpha F_1(z,\ z_0)-\beta\left(\frac{2i}{z_0-\bar{z}_0}\right)^s F_2(z,\ z_0) \tag{3.7}$$

which in the hydrodynamics of layers of thickness $h = y^s$, $s > 0$ can be treated as the complex potential of a vortex source. Carrying out n-fold Σ-differentiation of (3.7) we obtain solutions in the form of negative formal powers

$$Z^{(-n)}(\alpha,\ \beta,\ z,\ z_0)=\frac{(-1)^{n-1}}{(n-1)!}\frac{d_\Sigma^n}{dz^n}L(\alpha,\ \beta,\ z,\ z_0) \tag{3.8}$$

which, at the point z_0, have poles of order n (here and henceforth, unless otherwise stated, $n = 1, 2, 3, \ldots$). Then, the complex potentials W_n, describing multipoles of order $2n$ with moments M_n, will be

$$W_n=\frac{M_n}{2\pi}(-1)^{n-1}(n-1)!Z^{(-n)}(\alpha,\ \beta,\ z,\ z_0) \tag{3.9}$$

In particular, from (3.7)–(3.9) for $n = 1$ and $\alpha = \cos\theta_1$, $\beta = \sin\theta_1$ we have the complex potential of a dipole with moment M_1, directed at an angle θ_1 to the x axis

$$W_1=\frac{M_1 s(yy_0)^{-s/2}}{2\pi R_1^2}\left\{e^{i\theta_1}(Q_{s/2-1}-Q_{s/2})\left[\frac{2yy_0(x-x_0-i(y-y_0))}{R^2}+iy_0\right]+\right.$$
$$\left.+(Q_{s/2-1}\cos\theta_1-iQ_{s/2}\sin\theta_1)[x-x_0-i(y+y_0)]\right\} \tag{3.10}$$

$$R_1=\sqrt{(x-x_0)^2+(y+y_0)^2},\quad Q_\nu=Q_\nu(1+2\varepsilon)$$

We will consider the special case when the multipoles are situated at the origin of coordinates ($z_0 = 0$). Writing the asymptotic expansion of the Legendre function $Q_{s/2}(1 + 2\varepsilon)$ as $|z_0| \to 0$, in accordance with formula (3.9.1(21)) in [9], we have in the limit

$$\Phi_1(z,0) = -\frac{r^{-s}}{k(s)}, \quad r = \sqrt{x^2+y^2}, \quad k(s) = \Gamma\left(\frac{s+1}{2}\right)\left[\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\right]^{-1}$$

Then, assuming $\alpha = k(s)/n$, $\beta = 0$ in (3.7), we obtain from (3.8)

$$Z^{(-n)}(z) = \frac{(-1)^n}{n!}\left[\frac{\partial^n}{\partial x^n}r^{-s} + isy\frac{\partial^{n-1}}{\partial x^{n-1}}r^{-s-2}\right] \tag{3.11}$$

After reduction and introducing the Gegenbauer polynomials $C_n^\lambda(\cos\theta)$ (θ is the angle of a polar system of coordinates) and using generating functions (10.10(29)) from [10] for them, the powers (3.11) can be written in the form

$$Z^{(-n)}(z) = r^{-n-s}\left[C_n^{s/2}(\cos\theta) - i\frac{s\sin\theta}{n}C_{n-1}^{s/2+1}(\cos\theta)\right] \tag{3.12}$$

Following [3, 7], we will call the positive formal power $Z^{(n)}(\alpha, \beta, z, z_0)$ the complex potential W_n, which is obtained by n-fold Σ-integration from z_0 to z of the generalized complex constant $\alpha + i\beta[2i/(z-\bar{z})]^s$ and by multiplication by $n!$, i.e.

$$Z^{(0)}(\alpha,\beta,z,z_0) = \alpha + i\beta\left(\frac{2i}{z-\bar{z}}\right)^s, \quad Z^{(n)}(\alpha,\beta,z,z_0) = n\int_{z_0}^{z} Z^{(n-1)}(\alpha,\beta,\zeta,z_0)d_\Sigma\zeta \tag{3.13}$$

The powers (3.13) for $z = z_0$ have zeros of order n, while at infinity they have poles of the same order. For $s = 0$ they take the form of analytic functions $(\alpha + i\beta)(z-z_0)^n$. Hence, by analogy with the plane-parallel case ($s = 0$), these powers can be regarded as complex potentials of multipoles of order $2n$, situated at infinity.

The form of the powers (3.13) is determined by the choice of α, β and z_0. In particular, when $\alpha = 1$, $\beta = 0$, $z_0 = 0$ they can be written most simply as

$$Z^{(0)}(z) = 1, \quad Z^{(n)}(z) = r^n\left[C_n^{s/2}(\cos\theta) + \frac{s\sin\theta}{n+s}C_{n-1}^{s/2+1}(\cos\theta)\right] \tag{3.14}$$

Note that the powers (3.12) and (3.14) are related to one another by a transformation of the inversion (2.1).

The well-known solutions [5, 11] of Eq. (1.6) follow from (3.8) and (3.13), in particular, when $s = 1$.

The systems of functions obtained in the form of the powers $Z^{(\pm n)}(\alpha, \beta, z, z_0)$ enable a number of methods of solving boundary value problems of processes described by Eqs (1.1) to be developed.

4. One of the methods rests on a generalization of the Cauchy integral, known for analytic functions. Using the second Green's formula for Eqs (1.1) written in x, y coordinates with $P = y^s$, $s > 0$, on the basis of the solutions (3.2) and (3.4), by analogy with the approach used previously in [7], we obtain the generalized Cauchy formula

$$\frac{1}{2\pi i}\int_C \Omega_-(z,\zeta)W(\zeta)d\dot{\zeta} - \Omega_+(z,\zeta)\overline{W(\zeta)}d\bar{\zeta} = \begin{cases} W(z), & z \in D \\ 0, & z \overline{\in} D \end{cases} \tag{4.1}$$

which defines the function $W(z)$ in terms of the value of the boundary C of region D. The integral in (4.1) is the generalized Cauchy integral. Its kernel $\Omega_\pm(z, \zeta)$ can be expressed in terms of the normalized ($M_1 = 2\pi$) complex potentials w_1 and w_1' of the dipoles (3.10) with moments along the x axis ($\theta_1 = 0$) and perpendicular to the x axis ($\theta_1 = \pi/2$)

$$\Omega_\pm(z,\zeta) = \frac{1}{2}\left(\frac{\zeta-\bar{\zeta}}{2i}\right)^s(w_1 \pm iw_1')$$

and have the form

$$\Omega_{\pm}(z,\zeta)=\frac{s}{2}\left(\frac{\zeta-\bar{\zeta}}{z-\bar{z}}\right)^{s/2}\frac{Q_{s/2-1}\pm Q_{s/2}}{z-\zeta_{\pm}},\quad \zeta_{+}=\bar{\zeta},\ \zeta_{-}=\zeta \tag{4.2}$$

$$Q_{\nu}=Q_{\nu}(1+2\varepsilon),\quad \varepsilon=-\frac{(z-\bar{\zeta})(\bar{z}-\bar{\zeta})}{(z-\bar{z})(\zeta-\bar{\zeta})}$$

From (4.2) with $s = 1$ we obtain expressions for the kernels, obtained previously in [7, 11]. For $s = 0$ we have $\Omega_{\pm}(z, \zeta) = \pm(z-\zeta_{\pm})^{-1}$, and (4.1) reduces to the Cauchy formula for an analytic function in the region D symmetrical about the x axis.

Replacing $W(\zeta)$ in (4.1) by the function $f(\zeta)$, continuous along the line C, we obtain a generalized Cauchy-type integral, which defines the function $W(z)$ in the region D, which satisfies Eq. (1.6).

5. Another method of solving boundary-value problems consists of expanding the function $W(z)$ in series in formal powers, which turns out to be possible for solving equations of the form (1.6) [1, 3].

We can represent the function $W(z)$ in a circle of radius a by a generalized Taylor series in powers of (3.13), i.e.

$$W(z)=\sum_{n=0}^{\infty} Z^{(n)}(\alpha_n,\beta_n,z,z_0),\quad |z-z_0|<a \tag{5.1}$$

The real coefficients α_n, β_n are found from the equations

$$\alpha_0+i\beta_0\left(\frac{2i}{z_0-\bar{z}_0}\right)^{s}=W(z_0),\quad n!\left[\alpha_n+i\beta_n\left(\frac{2i}{z_0-\bar{z}_0}\right)^{s}\right]=W_{\Sigma}^{[n]}(z)|_{z=z_0}$$

which are obtained by n-fold Σ-differentiation of this series.

At the end $a_1 < |z-z_0| < a_2$ the function $W(z)$ can be represented in the form of a generalized Laurent series in powers of (3.8) and (3.13)

$$W(z)=\sum_{n=0}^{\infty} Z^{(n)}(\alpha_n,\beta_n,z,z_0)+\sum_{n=1}^{\infty} Z^{(-n)}(\alpha_{-n},\beta_{-n},z,z_0)$$

After introducing the residue of the function $W(z)$ as the Σ-integral (2.3), divided by $2\pi i$, the fundamental theorem of the residues of this function can be formulated and proved in the same way as in the case $s = 1$ [7].

Note that for the functions $W(z)$ we have an analogue of Sokhotskii's formulae, a theorem on analytic continuation, the mirror-image principle and other properties of the class of functions [1, 2], to which $W(z)$ belongs.

Hence, the fundamental theories of the functions $W(z)$ described above are a complete analogue of the basic ideas of analytic functions, where the analytic functions are a special case of the functions $W(z)$ when $s = 0$. This theory enables boundary-value problems to be solved in final form for two-dimensional processes.

6. We will employ the expansion in a generalized series (5.1) to find systems of functions which are used to solve boundary-value problems. That is, by setting up in the semicircle $z = e^{i\theta}$, $\theta \in [0, \pi]$ a series of powers of (3.14), we obtain the function

$$E(\alpha,z)=\sum_{n=0}^{\infty}\frac{(\alpha r)^n}{(s)_n}\left[C_n^{s/2}(\cos\theta)+i\frac{s\sin\theta}{n+s}C_{n-1}^{s/2-1}(\cos\theta)\right]$$

where $(s)_n$ is the Pochhammer symbol [9]. Using formula (10.9(30)) from [10], we obtain after reduction

$$E(\alpha,z)=\Gamma(\nu+1)e^{\alpha x}\left(\frac{\alpha y}{2}\right)^{-\nu}[J_{\nu}(\alpha y)+iJ_{\nu+1}(\alpha y)],\quad \nu=\frac{s-1}{2} \tag{6.1}$$

Taking into account the property of Bessel functions [10] $J_\nu(-\alpha y) = (-1)^\nu J_\nu(\alpha y)$, on the basis of (6.1) we introduce the functions

$$\mathrm{Ch}(\alpha, z) = \frac{E(\alpha, z) + E(-\alpha, z)}{2}, \quad \mathrm{Sh}(\alpha, z) = \frac{E(\alpha, z) - E(-\alpha, z)}{2} \tag{6.2}$$

Bearing in mind the property of Bessel functions of imaginary argument [10] $J_\nu(i\alpha y) = i^\nu I_\nu(\alpha y)$, we have

$$\mathrm{Ch}(i\alpha, z) = C(\alpha, z), \quad \mathrm{Sh}(i\alpha, z) = iS(\alpha, z)$$

where

$$C(\alpha, z) = \Gamma(\nu+1)\left(\frac{\alpha y}{2}\right)^{-\nu} [\cos \alpha x I_\nu(\alpha y) - i \sin \alpha x I_{\nu+1}(\alpha y)] \tag{6.3}$$

$$S(\alpha, z) = \Gamma(\nu+1)\left(\frac{\alpha y}{2}\right)^{-\nu} [\sin \alpha x I_\nu(\alpha y) + i \cos \alpha x I_{\nu+1}(\alpha y)]$$

Using the formulae for the differentiation of Bessel functions, it can be shown that Eqs (1.1) in x, y coordinates when $P = y^s$, $s > 0$ are satisfied by the functions

$$\varphi = \Gamma(\nu+1)\left(\frac{\alpha y}{2}\right)^{-\nu} I_\nu(\alpha y)\begin{cases}\cos \alpha x \\ \sin \alpha x\end{cases}, \quad \psi = \Gamma(\nu+1)\left(\frac{\alpha}{2}\right)^{-\nu} y^{\nu+1} I_{\nu+1}(\alpha y)\begin{cases}-\sin \alpha x \\ \cos \alpha x\end{cases}$$

which follow from (1.2) and (6.3). Then, the functions (6.3) are solution of Eq. (1.6).

In particular, when $s = 0$ ($\nu = -1/2$), the functions (6.1)–(6.3) take the form of analytic functions: $e^{\alpha z}$, ch αz, sh αz, cos αz, sin αz. Consequently, (6.1)–(6.3) can be considered as analogues of the corresponding analytic functions when $s > 0$.

Using the differential properties of Bessel functions [10], we can obtain solutions of Eq. (1.6) independently of (6.1)–(6.3). For example, if we replace the Bessel functions I_ν and $I_{\nu+1}$ by MacDonald functions K_ν and $-K_{\nu+1}$ in the solutions (6.3), we obtain other solutions of Eq. (1.6). We will introduce the functions $T(\alpha, z)$ and $T_1(\alpha, z)$ as linear superpositions of the functions (6.3) and these solutions

$$T(\alpha, z) = y^{-\nu}[(A\cos \alpha x + B\sin \alpha x) I_\nu(\alpha y) - i(A\sin \alpha x - B\cos \alpha x) I_{\nu+1}(\alpha y)] \tag{6.4}$$

$$T_1(\alpha, z) = y^{-\nu}[(A\cos \alpha x + B\sin \alpha x) K_\nu(\alpha y) + i(A\sin \alpha x - B\cos \alpha x \ \ I_{\nu+1}(\alpha y)]$$

where A and B are real constants.

The general solution $W(z)$ of Eq. (1.6) can be represented as the sum of $T(\alpha, z)$ and $T_1(\alpha, z)$. In particular (as follows from the asymptotic for of the functions I_ν, K_ν [10]), if the solution $W(z)$ is bounded when $y = 0$, it can be expressed in terms of the function $T(\alpha, z)$; if the solution $W(z)$ is bounded at infinity, it can be represented in terms of $T_1(\alpha, z)$.

7. We will use the apparatus of the functions $W(z)$ to solve boundary-value conjugation problems for two-dimensional processes in an inhomogeneous layer. The variable conductivity of the layer is due to a change in its permeability k and thickness h. To fix our ideas we will assume that h varies continuously while k varies in jumps along a certain curve L' of the ζ plane of the base of the layer. Suppose this curve is the boundary of regions D'_1 and D'_2, the permeabilities of the media of which k_1 and k_2 and the processes in them are described by the following complex potentials

$$W_j = k_j\varphi_j + i\psi_j/h, \ j = 1, 2 \tag{7.1}$$

We will assume that the conjugation conditions $\varphi_1 = \varphi_2$, $\psi_1 = \psi_2$ are satisfied on L'. These conditions, for example, in the case of seepage [4, 12], express the continuity of the pressure and the flow rate of the liquid. These conditions can be written as follows in terms of the complex potentials (6.1)

$$(1-\lambda)W_1 = W_2 + \lambda\overline{W_2} \quad \left(\lambda = \frac{k_1 - k_2}{k_1 + k_2}, \quad \lambda \in [-1,1]\right) \tag{7.2}$$

Suppose the process in the layer with constant permeability k_0, taken as the unit of measurement of the permeability of the media ($k_0 = 1$), is described by the complex potential $W_0 = \varphi_0 + i\psi_0/h$. We will represent it in the form $W_0 = W_{01} + W_{02}$, where the functions W_{01} and W_{02} have singular points in the regions D'_1 and D'_2.

In the ζ plane it is required to solve the boundary-value conjugation problem (1.3), (7.2) for the function $W(\zeta)$. A similar form of the conjugation problem has been investigated in a number of publications (see the review in [13]) by reducing it to the corresponding generalized problem in the class of analytic functions, which, in the final analysis, leads to the need to solve an integral equation with conditions which are more complex than (7.2) and which therefore gives rise to greater difficulties.

Using the conformal covariance of problem (1.3), (7.2), we reduce it to the simpler (canonical) conjugation problem (1.6), (7.2) for the function $W(z)$ with boundary L of the regions D_1 and D_2 in the half-plane $\operatorname{Im} z \geqslant 0$. We will consider the case when the boundaries L differ considerably and they can model a semicircle and straight lines, for which solutions of the canonical problem can be obtained in closed form.

We will assume that a semicircle of radius a is drawn from the origin of coordinates: $z = ae^{i\theta}$, $\theta \in [0, \pi]$ is the boundary of regions $D_1(|z| > a)$ and $D_2(|z| < a)$. Suppose $|W_{01}(z)| = O(|z|^{\mu_1})$ as $|z| \to 0$ and $|W_{02}(z)| = O(|z|^{-\mu_2})$ as $|z| \to \infty$ ($\mu_1 = \mu - s(1+\lambda)/2$, $\mu_2 = \mu + s(1-\lambda)/2$, $\mu > 0$). Then, we have the complex potentials of the process in regions D_1 and D_2

$$\begin{aligned}
W_1 &= W_{01}(z) + \left(\frac{a}{|z|}\right)^s \left[\lambda \overline{W_{01}\left(\frac{a^2}{\bar z}\right)} + G_1\left(\frac{a^2}{\bar z}\right) + \lambda \overline{G_1\left(\frac{a^2}{\bar z}\right)}\right] + (1+\lambda)[W_{02}(z) + G_2(z)] \\
W_2 &= W_{02}(z) + \left(\frac{a}{|z|}\right)^s \left[\lambda \overline{W_{02}\left(\frac{a^2}{\bar z}\right)} + G_2\left(\frac{a^2}{\bar z}\right) - \lambda \overline{G_2\left(\frac{a^2}{\bar z}\right)}\right] + (1-\lambda)[W_{01}(z) + G_1(z)] \\
G_j(z) &= \varkappa_j \int_{p_j}^{1} \tau^{s/2-1-\varkappa_j} W_{0j}(z\tau)\,d\tau, \quad \varkappa_j = (-1)^j \frac{\lambda s}{2}, \quad j = 1,2, \quad p_1 = 0, \quad p_2 = \infty
\end{aligned} \tag{7.3}$$

In fact, formulae (7.3) are the solution of the problem. The integrals, that define the functions $G_j(z)$, by virtue of the above-mentioned limitations on $W_{0j}(z)$ converge, and these functions satisfy Eq. (1.6). The expressions in square brackets, multiplied by $(a/|z|)^s$, according to Eq. (2.1), also satisfy this equation. Consequently, (7.3) is a solution of Eq. (1.6). Since on the boundary $L: z = ae^{i\theta}$, $\theta \in [0, \pi]$ we have the equation $z\bar z = a^2$, it is easy to show that solution (7.3) satisfies condition (7.2) on it.

Suppose the half-line $L: x = 0, y \geqslant 0$ is the boundary of the regions $D_1(x > 0)$ and $D_2(x < 0)$. Then, the process in these regions is described by the complex potentials

$$W_1 = W_0(z) + \lambda[\overline{W_{01}(-\bar z)} + W_{02}(z)], \quad W_2 = W_0(z) - \lambda[W_{01}(z) + \overline{W_{02}(-\bar z)}] \tag{7.4}$$

which are a solution of the problem.

In fact, $\overline{W_{01}(-\bar z)} = \overline{W}_{01}(-x, y)$ and $\overline{W_{02}(-\bar z)} = \overline{W}_{02}(-x, y)$ have singular points in the regions D_2 and D_1 and satisfy Eq. (1.6). Then, (7.4) are solutions of Eq. (1.6). It can be shown that condition (7.2) is satisfied for this boundary L.

Suppose now that the straight line L: $y = b = \text{const}$ is a boundary of the region $D_1(y > b)$ and $D_2(y < b)$. We will seek a solution of the problem in the form [14]

$$W_1 = W_0 + \int_0^\infty \Lambda_1(\alpha) T_1(\alpha, z)\,d\alpha, \quad W_2 = W_0 + \int_0^\infty \Lambda_2(\alpha) T(\alpha, z)\,d\alpha \tag{7.5}$$

where $T(\alpha, z)$ and $T_1(\alpha, z)$ are defined by (6.4). We will assume that the function $\varphi_0(x, y) = \operatorname{Re} W_0$ on the boundary $L: y = b$ is continuous and absolutely integrable with respect to $x \in]-\infty, \infty[$. Consequently, it can be expanded in a Fourier integral [15]

$$\varphi_0(x,b) = \int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]d\alpha \tag{7.6}$$

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_0(t,b)\cos\alpha t dt, \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_0(t,b)\sin\alpha t dt$$

Then, satisfying conditions (7.2) on $L: y = b$ and taking the equation $I\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = z^{-1}$ into account [10], as well as Eqs (6.4), (7.5) and (7.6), we have

$$\begin{aligned}
W_1 &= W_0 + y^{-\nu}\int_0^\infty \Lambda(\alpha)I_{\nu+1}(\alpha b)[K_\nu(\alpha y)\omega(\alpha,x) + iK_{\nu+1}(\alpha y)\omega_1(\alpha,x)]d\alpha \\
W_2 &= W_0 - y^{-\nu}\int_0^\infty \Lambda(\alpha)K_{\nu+1}(\alpha b)[I_\nu(\alpha y)\omega(\alpha,x) - iI_{\nu+1}(\alpha y)\omega_1(\alpha,x)]d\alpha \\
\Lambda(\alpha) &= 2\lambda b^\nu[(1-\lambda)(\alpha b)^{-1} + 2\lambda I_\nu(\alpha b)K_{\nu+1}(\alpha b)]^{-1} \\
\omega(\alpha,x) &= A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x = \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_0(t,b)\cos\alpha(t-x)dt \\
\omega_1(\alpha,x) &= \frac{1}{\alpha}\frac{\partial\omega(\alpha,x)}{\partial x} = \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_0(t,b)\sin\alpha(t-x)dt
\end{aligned} \tag{7.7}$$

The integrals in (7.7) converge.

In fact, the moduli M_1 and M_2 of the integrands in the complex potentials W_1 and W_2 are functions of $\alpha \in [0, \infty[$ with a limited change in each finite interval. Since, as $\alpha \to \infty$, by virtue of the asymptotic form of the functions I_ν, K_ν [10], we have $M_1 = O(\sqrt{(b/y)}e^{-\alpha(y-b)})$ and $M_2 = O(\sqrt{(b/y)}e^{-\alpha(b-y)})$, then in the limit M_1 and M_2 take finite values. Consequently, the integrals in (7.7) converge, and these formulae give the solution of the problem.

Note that solution (7.7) also occurs when in the regions $D_1(y > b)$ and $D_2(y < b)$ the laws of variation of the layer thickness are different: $h_i = (y/b)^{s_j}$, $s_j > 0$, if in the complex potentials W_j we take ν to mean $(s_j - 1)/2$, $j = 1, 2$.

Solutions (7.3), (7.4) and (7.7) define, in particular, the complex potentials of seepage processes when one of the regions D_1 or D_2 is impenetrable ($k_1 = 0$, $\lambda = -1$ or $k_2 = 0$, $\lambda = 1$), or is a cavern ($k_1 = \infty$, $\lambda = 1$ or $k_2 = \infty$, $\lambda = -1$). These solutions take a well-known form for plane-parallel processes ($s = 0$) [6] and axisymmetric processes ($s = 1$) [7, 14].

Note that solutions (7.3), (7.4) and (7.7) describe arbitrary two-dimensional processes modelled by singular points of the function $W_0(z)$. In particular, these solutions define the process caused by a source (sink), if we take $W_0(z)$ in them to mean the complex potential (3.5).

Applying conformal transformations to (7.3), (.4) and (7.7), we obtain, in a closed form, a wide class of boundary-value conjugation problems (1.3) and (7.2) for two-dimensional processes in inhomogeneous layers, the conductivity of which is modelled by relation (1.5).

The boundary-value problems investigated above using the developed apparatus of functions $W(z)$ does not exhaust its possibilities. It can be used for a wide range of two-dimensional processes of different physical kinds, described by Eqs (1:1). In particular, for seepage in layers of conductivity (1.5) with $s = 1$ a solution of a number of conjugation problems is obtained [5, 14]. This apparatus may also be of interest when investigating plane-parallel flows of an ideal compressible fluid, defined by Chaplygin's equations.

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